CONVERGENCE IN MEASURE
MATH 891

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Abstract. This short note accompanies an introductory talk on the convergence in measure of a sequence of measurable functions.

Contents

1. Introduction

Throughout, let $(X, \mathscr{M}, \mu)$ be a measure space. Let $(f_n : X \to \mathbb{C})_{n\geq 1}$ be a sequence of measurable functions and $f : X \to \mathbb{C}$ be a measurable function.

Definition 1.1. We say that $(f_n)_{n\geq 1}$ *converges in measure* to f if, for all $\varepsilon > 0$, there exists $N > 0$ such that

$$\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$$

for all $n \geq N$.

We make a few observations about this definition of convergence. First, the set $A_{\varepsilon,n} \coloneqq \{x \in X\colon |f_n(x) - f(x)| > \varepsilon\}$ is in fact measurable for any $\varepsilon > 0$ and $n \in \mathbb{N}$. Indeed, first note that we can see $g_n(x) \coloneqq |f_n(x)-f(x)|$ as a composition of a measurable functions followed by a continuous function: the absolute value function. Thus the function g_n is measurable. All that's left to see is that $A_{\varepsilon,n}$ is precisely $g_n^{-1}((\varepsilon, \infty))$ which is a measurable set. Second, convergence in measure is a weaker notion of uniform convergence. Indeed, we have the following proposition.

Proposition 1.2. *If $(f_n)_{n\geq 1}$ converges uniformly to f, then $(f_n)_{n\geq 1}$ converges in measure to f.*

Proof. Let $\varepsilon > 0$ be arbitrary. We can use uniform convergence to select $N > 0$ for which $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and all $x \in X$. But then this means that the set of $x \in X$ for which $|f_n(x) - f(x)| > \varepsilon$ is empty,

Date: November 26, 2025.

and thus has measure zero. But this automatically satisfies the condition in Definition 1.1 and thus $(f_n)_{n\geq 1}$ converges in measure to f . □

The converse of Proposition 1.2 is false; see Section 3. If we want to loosen the hypothesis of uniform convergence to simply pointwise convergence, we need our measure space $(X, \mathscr{M}, \mu)$ to be a *finite* measure space. Again, we refer the reader to Section 3 for a counter-example in the general setting.

It can be useful, as an intuition for what convergence in measure is, to keep in mind the following equivalent notion of convergence in measure.

Proposition 1.3. *The sequence $(f_n)_{n\geq 1}$ converges in measure to f if and only if, for all $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

Proof. The reverse implication is clear.

Assume now that $(f_n)_{n\geq 1}$ converges to f in measure. Hence, for all $\varepsilon > 0$, there exists $N > 0$ such that $\mu(A_{\varepsilon,n}) < \varepsilon$ for all $n \geq N$. Let $\varepsilon' > 0$. If $\varepsilon \leq \varepsilon'$, then we have that $\mu(A_{\varepsilon,n}) < \varepsilon < \varepsilon'$. Suppose that $\varepsilon > \varepsilon'$. Then, there exists $N' > 0$ such that, for all $n \geq N'$, we have $\mu(A_{\varepsilon,n}) \leq \mu(A_{\varepsilon',n}) < \varepsilon'$. □

The main concern of this report is to prove the following collection of results, which can be found in [Fol99, Section 2.4], [RF10, Chapter 18.2, Exercise 15], and [Rud87, Chapter 3, Exercise 18].

Theorem 1.4. *Assume that $\mu(X) < \infty$.*

(a) If $(f_n)_{n\geq 1}$ converges to f almost everywhere, then $(f_n)_{n\geq 1}$ converges to f in measure.
(b) Let $1 \leq p \leq \infty$. Suppose that $f_n \in L^p(\mu)$ for all $n \geq 1$ and that $\|f_n - f\|_p$ converges to zero. Then, $(f_n)_{n\geq 1}$ converges to f in measure.
(c) If $(f_n)_{n\geq 1}$ converges to f in measure, then $(f_n)_{n\geq 1}$ has a subsequence converging to f almost everywhere.

Before proving Theorem 1.4, we state and prove the following well-known result (see, e.g., [Rud87, Chapter 3, Exercise 16] and [Fol99, Theorem 2.33]). We will use it in proving part (a) of Theorem 1.4.

Egorov's Theorem. *Assume that $\mu(X) < \infty$. Suppose that $(f_n)_{n\geq 1}$ converges to f pointwise. Let $\varepsilon > 0$. Then, there is $E \in \mathscr{M}$ with $\mu(E^c) < \varepsilon$ such that $(f_n)_{n\geq 1}$ converges uniformly to f on E.*

Proof. For $N, k > 0$, we define

$$S(N,k) := \bigcap_{n\geq N} \{x \in X : |f_n(x) - f(x)| < \frac{1}{k}\}.$$

Since $S(N,k) \subseteq S(N+1,k)$, by continuity of measure from below we have that

$$\lim_{N\to\infty} \mu(S(N,k)) = \mu(\bigcup_{N\geq 1} S(N,k)).$$

Let $x \in X$ and $k > 0$. Since $(f_n)_{n\geq 1}$ converges to f, there exists $N > 0$ such that $|f_n(x) - f(x)| < \frac{1}{k}$ for all $n \geq N$. Thus, $x \in \cup_{N\geq 1} S(N,k)$. It follows that $\lim_{N\to\infty} \mu(S(N,k)) = \mu(X)$. Let $\varepsilon > 0$. For all $k > 0$, the sequence

$(\mu(S(N,k)))_{N\geq 1}$ is non-decreasing and converges to $\mu(X)$. So, there exists N_k such that $\mu(S(N_k,k)) + \frac{\varepsilon}{2^k} > \mu(X)$. Define $E := \cap_{k\geq 1} S(N_k,k)$.

Obviously, $E \in \mathscr{M}$. Furthermore,

$$\mu(E^c) = \mu(\bigcup_{k\geq 1}(X \setminus S(N_k,k))) \leq \sum_{k\geq 1}(\mu(X) - \mu(S(N_k,k))) < \varepsilon \sum_{k\geq 1} \frac{1}{2^k} = \varepsilon.$$

It remains to show that $(f_n)_{n\geq 1}$ converges uniformly to f on E. Let $\varepsilon > 0$. Let k be such that $\frac{1}{k} < \varepsilon$. For all $n \geq N_k$, we have that $|f_n(x) - f(x)| < \frac{1}{k} < \varepsilon$ as x is in $S(N_k,k)$. We conclude that $(f_n)_{n\geq 1}$ converges uniformly to f on E. □

2. Proof of Theorem 1.4

2.1. Proof of Part (a). Let $A \in \mathscr{M}$ be such that $(f_n)_{n\geq 1}$ converges to f on A and $\mu(A^c) = 0$. Let $\varepsilon > 0$. Denote by $\mathscr{M}|_A$ the σ-algebra consisting of sets of the form $F \cap A$ for $F \in \mathscr{M}$ and by $\mu|_A$ the restriction of μ on this new σ-algebra. By Egorov's Theorem, there exists $E \in \mathscr{M}|_A$ such that $\mu|_A(A \setminus E) < \varepsilon$, and where $(f_n)_{n\geq 1}$ converges to f uniformly on E. Let N be a constant from the uniform convergence. Let $n \geq N$. We have

$$\begin{aligned}
&\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) \\
&= \mu(\{x \in A : |f_n(x) - f(x)| > \varepsilon\}) + \mu(\{x \in A^c : |f_n(x) - f(x)| > \varepsilon\}) \\
&= \mu|_A(\{x \in A : |f_n(x) - f(x)| > \varepsilon\}) \\
&= \mu|_A(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) + \mu|_A(\{x \in A \setminus E : |f_n(x) - f(x)| > \varepsilon\}) \\
&= \mu|_A(\{x \in A \setminus E : |f_n(x) - f(x)| > \varepsilon\}) \\
&< \varepsilon
\end{aligned}$$

which implies that $(f_n)_{n\geq 1}$ converges to f in measure. This concludes the proof of part (a).

2.2. Proof of Part (b). Suppose first that $p < \infty$. Let $\varepsilon > 0$ be arbitrary. Recall that we defined

$$A_{\varepsilon,n} := \{x \in X \colon |f_n(x) - f| > \varepsilon\}$$

for $n \geq 1$. Then each $A_{\varepsilon,n}$ is measurable and since $A_{\varepsilon,n} \subseteq X$ we have that

$$\int_X |f_n - f|^p \, d\mu \geq \int_{A_{\varepsilon,n}} |f_n - f|^p \, d\mu. \tag{1}$$

On the other hand, by definition $|f_n - f|^p > \varepsilon^p$ on $A_{\varepsilon,n}$. Thus

$$\int_{A_{\varepsilon,n}} |f_n - f|^p \, d\mu > \int_{A_{\varepsilon,n}} \varepsilon^p \, d\mu = \varepsilon^p \mu(A_{\varepsilon,n}). \tag{2}$$

Now select $N > 0$ for which $m \geq N$ implies $\|f_m - f\|_p < \varepsilon^{\frac{p+1}{p}}$. Then we deduce that for $m \geq N$,

$$\int_X |f_m - f|^p \, d\mu < \varepsilon^{1+p}. \tag{3}$$

By combining Equations (1) and (2), we find that Equation (3) implies that $\varepsilon^p \mu(A_{\varepsilon,m}) < \varepsilon^{1+p}$. In other words, $\mu(A_{\varepsilon,m}) < \varepsilon$ for all $m \geq N$. But this precisely means that $(f_n)_{n\geq 1}$ converges to f in measure.

We now do the case when $p = \infty$. Let $\varepsilon > 0$. Since $\|f_n - f\|_\infty \to 0$, there is a $N > 0$ such that $n \geq N$ implies that $\|f_n - f\|_\infty < \varepsilon$. In particular, ε is an essential bound, which implies that $\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) = 0$, and so $(f_n)_{n\geq 1}$ converges to f in measure.

2.3. **Proof of Part (c).** First, let $j \geq 1$. Using our hypothesis, we select $n_j > 0$ so that for $n \geq n_j$,

$$\mu\left(\{x \in X : |f_n(x) - f(x)| > 2^{-j}\}\right) < 2^{-j}. \tag{4}$$

Let $E_j := A_{2^{-j}, n_j} = \{x \in X : |f_{n_j}(x) - f(x)| > 2^{-j}\}$. Then by Equation (4), $\mu(E_j) < 2^{-j}$. Now let

$$\mathcal{F} := \limsup_j E_j = \bigcap_{k\geq 1} \bigcup_{j\geq k} E_j.$$

Notice that $\cup_{j\geq k} E_j \supseteq \cup_{j\geq k+1} E_j$: this is simply because $\cup_{j\geq k} E_j = \cup_{j\geq k+1} E_j \cup E_k$. Thus, by continuity from above (which we can use since each E_j has finite measure), we have that

$$\mu(\mathcal{F}) = \lim_{k\to\infty} \mu(\cup_{j\geq k} E_j) \leq \lim_{k\to\infty} \sum_{j\geq k} \mu(E_j) < \lim_{k\to\infty} \sum_{j\geq k} \frac{1}{2^j} = \lim_{k\to\infty} \frac{1}{2^{k-1}} = 0.$$

Thus $\mu(\mathcal{F}) = 0$. On the other hand, $x \in \mathcal{F}^c$ if and only if there is $k > 0$ such that for all $j \geq k$, we have $|f_{n_j}(x) - f(x)| \leq 2^{-j}$. In other words, $\lim_{j\to\infty} f_{n_j}(x) = f(x)$. But this means that the subsequence $(f_{n_j})_{j\geq 1}$ of $(f_n)_{n\geq 1}$ converges to f pointwise on $\mathcal{F}^c$. Since $\mu(\mathcal{F}) = 0$ this means $(f_{n_j})_{j\geq 1}$ converges to f almost everywhere, which completes the proof.

3. Beyond Theorem 1.4

In this section, we discuss some consequences of Theorem 1.4, analyze its reverse implications, and study it under weaker hypotheses.

Corollary 3.1. *If $(f_n)_{n\geq 1}$ converges pointwise to f, then $(f_n)_{n\geq 1}$ converges in measure to f.*

Proof. This is a trivial consequence of part (a) of Theorem 1.4. □

If $\mu(X) = \infty$, then Theorem 1.4 (a) is false, as shown in the next example.

Example 3.2. Consider the Lebesgue measure space $(\mathbb{R}, \mathfrak{M}, m)$. Then the sequence of functions $(f_n)_{n\geq 1}$ defined by $f_n := \mathbb{1}_{[n,n+1]}$ converges pointwise to $f \equiv 0$ on $\mathbb{R}$. However, the sequence fails to converge in measure to f. Indeed, for $0 < \varepsilon < 1$ and any $n \geq 1$, the set of $x \in \mathbb{R}$ for which $|f_n(x) - f(x)| > \varepsilon$ is precisely $[n, n+1]$. But $m([n, n+1]) = 1$ which dominates ε. So the sequence $(f_n)_{n\geq 1}$ *does not* converge in measure to f.

Note that the same example implies that Egorov's Theorem is false when $\mu(X) = \infty$.

The situation is different for Theorem 1.4 (b) and (c), as we have never used the hypothesis that X has finite measure.

Corollary 3.3. *Parts (b) and (c) of Theorem 1.4 hold when $\mu(X) = \infty$.*

We investigate the converse of Theorem 1.4. Most of the information can be found in [Fol99, Section 2.4]. First, the converse of Theorem 1.4 (a) is false. This can be seen (using part (b)) by the fact that a sequence of measurable functions can converge in $L^p(\mu)$ without converging almost everywhere.

Example 3.4. Let $Y := [0,1] \subseteq \mathbb{R}$. Consider the Lebesgue measure space $(Y, \mathfrak{M}, m)$ and define $f_n := \mathbb{1}_{[\frac{j}{2^k}, \frac{j+1}{2^k}]}$ where $n = 2^k + j$ with $0 \leq j < 2^k$. We have that

$$\int f_n dm = m([\frac{j}{2^k}, \frac{j+1}{2^k}]) = \frac{1}{2^k} \xrightarrow[n\to\infty]{} 0,$$

which implies that $(f_n)_{n\geq 1}$ converges in measure to 0. However, this sequence does not converge to 0 almost everywhere. Indeed, let $\varepsilon \in (0,1)$. For any $N > 0$, one can take $n = 2^k$ larger than N and obtain that $\{x \in \mathbb{R} : |f_n(x)| > \varepsilon\} = [0, \frac{1}{2^k}]$ has measure $\frac{1}{2^k} > 0$. Hence, around 0, there is always a set of positive measure on which $(f_n)_{n\geq 1}$ does not converge to 0.

The converse of Theorem 1.4 (b) is false.

Example 3.5. Let $Y := [0,1] \subseteq \mathbb{R}$. Consider the Lebesgue measure space $(Y, \mathfrak{M}, m)$ and define $f_n := n\mathbb{1}_{[0,\frac{1}{n}]}$ for $n \geq 1$. Let $\varepsilon > 0$. Let $N > 0$ be such that $\frac{1}{N} < \varepsilon$. Thus,

$$m(\{x \in \mathbb{R} : n\mathbb{1}_{[0,\frac{1}{n}]}(x) > \varepsilon\}) \leq \frac{1}{n} < \varepsilon$$

for all $n \geq N$ and so $(f_n)_{n\geq 1}$ converges to 0 in measure. However,

$$\int n^p \mathbb{1}_{[0,\frac{1}{n}]} dm = n^{p-1}$$

for all $p < \infty$, which implies that $\|f_n\|_p$ does not tend to 0 as $n \to \infty$. Hence, $(f_n)_{n\geq 1}$ does not converge to 0 in $L^p(m)$, for $1 \leq p < \infty$.

The same example also provides a counter-example for when $p = \infty$. Indeed, in that case, for all $n \geq 1$,

$$m(\{x \in \mathbb{R} : n\mathbb{1}_{[0,\frac{1}{n}]}(x) > M\}) = 0$$

happens if and only if $M \geq n$. Thus, $\|f_n\|_\infty = n$ for all $n \geq 1$, which does not converge to 0.

Finally, the converse of Theorem 1.4 (c) is obviously false:

Example 3.6. Let $Y := [0,1] \subseteq \mathbb{R}$. Consider the Lebesgue measure space $(Y, \mathfrak{M}, m)$ and define

$$f_n(x) := \begin{cases} x & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

for all $x \in \mathbb{R}$ and $n \geq 1$. Then, $(f_{2n})_{n\geq 1}$ converges pointwise to 0, but $(f_n)_{n\geq 1}$ does not converge in measure to 0. Indeed,

$$m(\{x \in X : |f_{2n+1}(x)| > \frac{1}{4}\}) = m((\frac{1}{4}, 1]) = \frac{3}{4} \not< \frac{1}{2}.$$

References

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