GROUP ACTIONS

1. Introduction

Definition 1.1. Let G be a group and X be a set. A **group action** of G on X is a map $\cdot : G \times X \to X$ such that
(1) $e \cdot x = x$, for all $x \in X$.
(2) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, for all $x \in X$ and $g_1, g_2 \in G$.

Theorem 1.2 (Characterization of group actions)**.** *Given a group action of G on a set X, let $\pi_g : X \to X$ be the map given by $\pi_g(x) = g \cdot x$. Then*
(a) For each $g \in G$, π_g is a permutation of X.
(b) The map $\rho : G \to S_X$ given by $\rho(g) = \pi_g$ is a group homomorphism.

Proof:
(a) Let $g \in G$. We claim that π_g^{-1} is $\pi_{g^{-1}}$. Indeed, note that for all $x \in X$ we have that

$$\pi_g(\pi_{g^{-1}}(x)) = \pi_g(g^{-1} \cdot x) = g \cdot (g^{-1} \cdot x) = (gg^{-1}) \cdot x = e \cdot x = x,$$

and

$$\pi_{g^{-1}}(\pi_g(x)) = \pi_{g^{-1}}(g \cdot x) = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = e \cdot x = x.$$

Hence $\pi_{g^{-1}}$ is a two sided inverse for π_g so that π_g is a bijection from X to X.
(b) First, note that by part (a) the map ρ is well defined. Let $g, h \in G$ be arbitrary. In order to show that $\rho(gh) = \rho(g) \circ \rho(h)$, we need to show that $\pi_{gh}(x) = \pi_g(\pi_h(x))$, for all $x \in X$. On one hand, we have that $\pi_{gh}(x) = (gh) \cdot x$, which is the same as $g \cdot (h \cdot x)$ by the second axiom of groups actions. But $g \cdot (h \cdot x) = \pi_g(\pi_h(x))$, and so $\pi_{gh}(x) = \pi_g(\pi_h(x))$.

Corollary 1.3. *Let a group G act on a set X. If $x \in X$, $g \in G$ and $y = g \cdot x$, then $x = g^{-1} \cdot y$. If $x \neq x'$ then $g \cdot x \neq g \cdot x'$.*

Proof: First, let $x \in X$ and $g \in G$. Let π_g be the permutation corresponding to g. Then $y = g \cdot x$ implies that $\pi_g(x) = y$. Since π_g is a bijection, $x = \pi_g^{-1}(y)$. Since $\pi_g^{-1} = \pi_{g^{-1}}$, this would imply that $x = \pi_{g^{-1}}(y)$, that is $x = g^{-1} \cdot y$. Now, let $x \neq x'$. Since π_g is a bijection, it is injective and so $\pi_g(x) \neq \pi_{g^{-1}}(x')$. That is, $g \cdot x \neq g \cdot x'$.

Example 1.4. The group D_4 acts on the set of 4 vertices of a regular square. Realizing r^k as a counterclockwise rotation by $\pi/2$ and $r^k s$ as a reflection across the line through the points $(-\cos(\pi k/4), -\sin(\pi k/4))$ and $(\cos(\pi k/4), \sin(\pi k/4))$, one can produce the following table of correspondences:

$s^l r^k \in D_4$	e	r	r^2	r^3	s	rs	$r^2 s$	$r^3 s$
$\sigma \in S_4$	ε	(1234)	(13)(24)	(1432)	(24)	(12)(34)	(13)	(14)(23)

Example 1.5 (Left Multiplication)**.** For a group G, the action of G on itself by left multiplication is a group action. Indeed, let $g, h, a \in G$. Then $(gh) \cdot a = (gh)(a) = (g)(ha) = g \cdot (h \cdot a)$, and $e \cdot a = ea = a$.

Example 1.6. For a group G, right multiplication of G on itself is not necessarily a group action. Indeed, let $g, h \in G$ and $x \in G$. Then $(gh) \cdot x = x(gh)$ however $g \cdot (h \cdot x) = xhg$. If G is abelain then the right multiplication map would be a group action, since $x(gh) = x(hg)$. However, we could instead define the action as $g \cdot x = xg^{-1}$. Then, $(gh) \cdot x = x(gh)^{-1} = xh^{-1}g^{-1}$, and $g \cdot (h \cdot x) = xh^{-1}g^{-1}$, and $e \cdot x = xe^{-1} = x$.

Example 1.7. The vector space $\mathbb{R}^n$ can act on itself by *translations*. To be more precise, we can define the action as $v \cdot w = v + w$ for vectors $v, w \in \mathbb{R}^n$. This is indeed a group action since for $v, v', w \in \mathbb{R}^n$, $v + (v' + w) = (v + v') + w$ and $0 + w = w$. It is easier to realize the permutations of the action of *translations* if we instead consider the group $\mathbb{Z}_4$. The action here is $[a] \cdot [b] = [a] + [b]$. Then the action of $[0]$ corresponds to the permutation ε. The action of $[1]$ corresponds to the permutation (1234). The action of $[2]$ corresponds to $(12)(34)$ and the action of $[3]$ corresponds to (1432).

Example 1.8 (Conjugation)**.** A group G may act on itself by *conjugation*: $g \cdot a = gag^{-1}$. This is indeed a group action since for all $g, h, x \in G$:

$$(gh) \cdot x = (gh)x(gh)^{-1} = ghxh^{-1}g^{-1}$$
$$g \cdot (h \cdot x) = g \cdot (hxh^{-1}) = ghxh^{-1}g^{-1}$$
$$e \cdot x = exe^{-1} = x.$$

Definition 1.9. A group action of G on X is called **faithful** if no two elements correspond to the same permutation: $(\forall g, h \in G, g \neq h, \exists x \in X \pi_g(x) \neq \pi_h(x))$

Theorem 1.10. *A group action of G on X is faithful if and only if the group homomorphism $\rho : G \to S_X$ is injective.*

Proof: Suppose G acts on X and the group action is faithful. Then for $g, h \in G$, $g \neq h$, $\pi_g(x) \neq \pi_h(x)$ for some $x \in X$. But then $\rho(g)(x) \neq \rho(h)(x)$, so that ρ is injective. Conversely, if ρ is injective, then for $g \neq h$, $\rho(g)(x) \neq \rho(h)(x)$, for some $x \in X$. But then $\pi_g(x) \neq \pi_h(x)$ so that the action of G on X is faithful.

Proposition 1.11. *The group action of G on itself by conjugation is faithful if and only if $Z(G) = \{e\}$.*

Proof: First, suppose that the action of G on itself by conjugation is faithful. Suppose that $Z(G)$ is not trivial. Choose $g \neq e \in Z(G)$. Note that for all $x \in G$, $\pi_g(x) = gxg^{-1} = gg^{-1}x = x$. But this is the same as the action of e, namely $\pi_e(x) = exe^{-1} = x$. So $\pi_g = \pi_e$, which contradicts our hypothesis that the action of conjugation is faithful. Conversely, suppose that $Z(G) = \{e\}$. Let $g, h \in G$ such that $g \neq h$. Suppose that the group action of G on itself by conjugation is not faithful. Then there exists $g, h \in G$, where $g \neq h$, such that $\pi_g(x) = \pi_h(x)$ for all $x \in G$. This would imply that

$$gxg^{-1} = hxh^{-1}$$
$$x(g^{-1}h) = (g^{-1}h)x.$$

But then $g^{-1}h \in Z(G)$. Since $Z(G) = \{e\}$, we have $g^{-1}h = e$, so that $g = h$, a contradiction.

2. Orbits and Stabilizers

Definition 1.12. Let a group G act on a set X. For each $x \in X$, be define the **orbit** of x to be

$$\mathrm{Orb}(x) = \{g \cdot x : g \in G\}.$$

We define the **stabilizer** of x to be

$$\mathrm{Stab}(x) = \{g \in G : g \cdot x = x\}.$$

Example 1.13. Let $GL_2(\mathbb{R})$ act on $\mathbb{R}^2$ by $T_A(x) = Ax$. Then $\mathrm{Orb}(0) = \{0\}$ and $\mathrm{Stab}(0) = GL_2(\mathbb{R})$. The orbit of $x = (1,0)^T$ is every non-zero vector in $\mathbb{R}^2$. Indeed, first note that for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \in GL_2(\mathbb{R})$, we must have that $a \neq 0$ or $c \neq 0$ since $\det(A) = 0$ otherwise. So the first column of A must be a non-zero vector. Hence $T_A(x) \neq 0$. Conversely, let $y = (a,c)^T \in \mathbb{R}^2$ be a non-zero vector. If $a \neq 0$, then $X = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}$ is invertible. Moreover, $T_X(x) = y$ and so $y \in \mathrm{Orb}(x)$. If $c \neq 0$ then $X = \begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix}$ is invertible. Moreover, $T_X(x) = y$ and so $y \in \mathrm{Orb}(x)$. The stabilizer is $\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}$, where $y \neq 0$. Indeed, let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Stab}(x)$. Then $T_X(x) = x$, and so $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which forces $d \neq 0$ (b can be a free variable). Conversely, any matrix $B = \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}$, where $y \neq 0$, is clearly invertible and in the stabilizer of x since $T_B(x) = x$.

Example 1.14. If we reconsider Example 1.13 this time with $GL_2(\mathbb{Z})$ acting on $\mathbb{Z}^2$, then the orbit of $x = (1,0)^T$ is *not* every non-zero vector in $\mathbb{Z}^2$. Indeed, note that $GL_2(\mathbb{Z})$ is the set of all matrices with entries in $\mathbb{Z}$ with determinant ± 1 (since the only units in $\mathbb{Z}$ are ± 1). Hence only such vectors with co-prime coordinates can be seen under the action of $GL_2(\mathbb{Z})$ on $\mathbb{Z}^2$. Indeed, consider a vector $y = (a,c)^T$ where $\gcd(a,c) \neq 1$. Suppose that there exists an invertible matrix $X = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in GL_2(\mathbb{Z})$ such that $T_X(x) = y$. This would imply that $\begin{pmatrix} a' \\ c' \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$, so $a' = a$ and $c' = c$. Since $X \in GL_2(\mathbb{Z})$, we have $ad' + c(-b)' = \pm 1$, which implies that $(\exists r, t \in \mathbb{Z}, ar + ct = 1)$. This is a characterization of co-prime integers, and so $\gcd(a,c) = 1$, which is a contradiction.

Example 1.15. Consider the group action of $\mathbb{R}$ on $\mathbb{R}^2$ by $t \cdot (x,y) = (x, y+t)$. This is indeed a group action since $0 \cdot (x,y) = (x, y+0) = (x,y)$ and for $r, t \in R$, $r \cdot (t \cdot (x,y)) = r \cdot (x, y+t) = (x, y+t+r)$ and $(r+t) \cdot (x,y) = (x, y+t+r)$. The orbits of the vectors $(x,y) \in \mathbb{R}^2$ can be seen geometrically as vertical lines through (x,y). The stabilizer for any vector $(x,y) \in \mathbb{R}^2$ is trivial (any other $0 \neq r \in \mathbb{R}$ would shift the point in some way).

Example 1.16. If we consider the action of D_4 on the set of vertices $X = \{1,2,3,4\}$, then the orbit of any vertex is X. Indeed, for a fixed vertex $x \in X$, each rigid motions r^k, $0 \leq k \leq 3$, sends x to $x + k \mod 4$. The stabilizer of vertices 1 and 3 are s and the stabilizer of vertices 2 and 4 are $r^2 s$ (Indeed, one can check the permutation table given in Example 1.4).

Definition 1.17. For a group G acting on a set X, we call $x \in X$ a **fixed point** for the action when $g \cdot x = x$ for all $g \in G$. Note that in this case, $\mathrm{Orb}(x) = \{x\}$ and $\mathrm{Stab}(x) = G$.

Example 1.18. Consider the action of G on itself by left conjugation. For a given $x \in G$, we have that $\operatorname{Orb}(x) = \{gxg^{-1} : g \in G\}$ which is the conjugacy class of x. Moreover, $\operatorname{Stab}(x) = \{g \in G : gxg^{-1} = x\}$ which is precisely the centralizer of x. We claim that x is a fixed point action if and only if $x \in Z(G)$. Indeed, suppose that x is a fixed point action. Then $g \cdot x = g$ for all $g \in G$, equivalently, $gx = xg$ for all $g \in G$. This would imply that $x \in Z(G)$. Conversely, if $x \in Z(G)$ then $xg = gx$ for all $g \in G$, so that $gxg^{-1} = x$ for all $g \in G$ making x a fixed point for the acting of conjugation.

Theorem 1.19. *Let G be a group which acts on a set X. Then the orbits of the action partition the set X.*

Proof: First, we prove that $\operatorname{Orb}(x) \neq \emptyset$ for every $x \in X$. This is trivial, since $e \cdot x = x \in \operatorname{Orb}(x)$ for all $x \in X$. Next, we need to show that for $x \neq y \in X$, if $\operatorname{Orb}(x) \neq \operatorname{Orb}(y)$ then $\operatorname{Orb}(x) \cap \operatorname{Orb}(y) = \emptyset$. Let $x, y \in X$ such that $x \neq y$. Assume towards a contradiction that $\operatorname{Orb}(x) \cap \operatorname{Orb}(y) \neq \emptyset$. Then there exists some $g, h \in G$ such that $g \cdot x = h \cdot y$. Let $w = h^{-1}g$ and $v = g^{-1}h$ be in G. We claim that $\operatorname{Orb}(x) \subseteq \operatorname{Orb}(y)$. Indeed, let $g' \cdot x \in \operatorname{Orb}(x)$ where $g' \in G$. Then

$$(g'v) \cdot y = g' \cdot (v \cdot y) = g' \cdot x.$$

Hence $g' \cdot x \in \operatorname{Orb}(y)$. We now claim that $\operatorname{Orb}(y) \subseteq \operatorname{Orb}(x)$. Indeed, let $h' \cdot y \in \operatorname{Orb}(y)$. Then

$$(h'w) \cdot x = h' \cdot (w \cdot x) = h' \cdot y.$$

Hence $h' \cdot y \in \operatorname{Orb}(x)$. This would imply that $\operatorname{Orb}(x) = \operatorname{Orb}(y)$, a contradiction. We conclude that $\operatorname{Orb}(x) \cap \operatorname{Orb}(y) = \emptyset$.

Theorem 1.20 (The Oribt-Stablizer Theorem)**.** *Let G be a group which acts on a set X. Then for all $x \in X$ we have that*

$$|G| = |\operatorname{Orb}(x)| \, |\operatorname{Stab}(x)| \, .$$

Proof: Let $x \in X$. We wish to show that $|G/\operatorname{Stab}(x)| = |\operatorname{Orb}(x)|$. Let $f : G/\operatorname{Stab}(x) \to \operatorname{Orb}(x)$ be the map defined by $f(g \operatorname{Stab}(x)) = g \cdot x$ (where $g \cdot x$ is the action of g on x). To show that this map is well defined, we must show that the image of any coset in $G/\operatorname{Stab}(x)$ under f doesn't depend on our choice of the representative. Indeed, suppose that $g' \operatorname{Stab}(x) = g \operatorname{Stab}(x)$ for some $g, g' \in G$. This would imply that $g^{-1}g' \in \operatorname{Stab}(x)$, which implies that $(g^{-1}g') \cdot x = x$, which then implies that $g' \cdot x = g \cdot x$. But then $f(g' \operatorname{Stab}(x)) = f(g \operatorname{Stab}(x))$, so that f is well defined. Now, we show that f is bijective. Indeed, to see that f is injective, suppose that $f(a \operatorname{Stab}(x)) = f(b \operatorname{Stab}(x))$ for some $a, b \in G$. This would imply that $a \cdot x = b \cdot x$ so that $(a^{-1}b) \cdot x = x$, but then $a^{-1}b \in \operatorname{Stab}(x)$ so that $a \operatorname{Stab}(x) = b \operatorname{Stab}(x)$. Finally, we show that f is surjective. Let $g \cdot x \in \operatorname{Orb}(x)$ for some $g \in G$. Choose $g \operatorname{Stab}(x) \in G/\operatorname{Stab}(x)$, so that $f(g \operatorname{Stab}(x)) = g \cdot x$.

We have thus established that $|G/\operatorname{Stab}(x)| = |\operatorname{Orb}(x)|$. By Lagrange's Theorem, we conclude that

$$|G| = |\operatorname{Stab}(x)| \, |G/\operatorname{Stab}(x)| = |\operatorname{Stab}(x)| \, |\operatorname{Orb}(x)| \, .$$

Definition 1.21. Let G be a group and $x \in G$. We define the **conjugacy class** of x to be the set

$$Cl(x) = \{gag^{-1} : g \in G\}.$$

We define the **centralizer** of x to be the set

$$C(x) = \{g \in G : gx = xg\}.$$

Theorem 1.22. *Let G be a finite group. Suppose that G acts on itself by conjugation. Let $x_1, \ldots, x_m$ be the representatives for the distinct orbits of the action that partition G. Then*

$$|G| = \sum_{i=1}^{m} [G : C(x_i)].$$

Proof: Note that when G is a finite group acting on itself by conjugation then $\mathrm{Orb}(x) = Cl(x)$ and $\mathrm{Stab}(x) = C(x)$ for each $x \in G$ (as in Example 1.18). In particular, as in the proof of the Oribt-Stablizer Theorem, we have that $|Cl(x)| = [G : C(x)]$, for each $x \in G$. In accordance to our hypothesis, $G = \sqcup_{i=1}^{m} \mathrm{Orb}(x_i)$. Therefore

$$|G| = \sum_{i=1}^{m} |\mathrm{Orb}(x_i)| = \sum_{i=1}^{m} |Cl(x_i)| = \sum_{i=1}^{m} [G : C(x_i)].$$

If we want, we can exlude those representatives from the list $x_1, \ldots, x_m$ that are fixed points in the action. By Example 1.18, these are precisley the elements that are in the center of $Z(G)$. Therefore, we may select $x_1, \ldots, x_k$ to be the representatives for the distinct orbits not contained in $Z(G)$ ($k \leq m$) so that

$$|G| = |Z(G)| + \sum_{i=1}^{k} [G : C(x_i)].$$