CURVES SEMINAR 2025 - YOUNG TABLEAUX

ABDULLAH ZUBAIR

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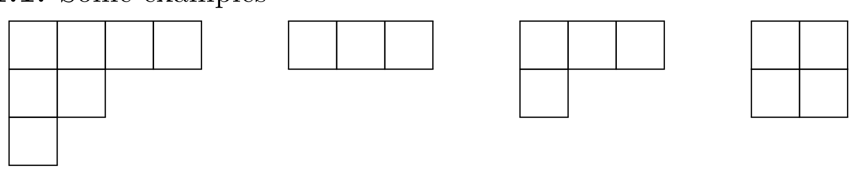
1. Lecture 1 - Wednesday September 17

1.1. Notation.

Definition 1: *A **young diagram** is a collection of boxes arranged in left justified rows, with a weakly decreasing number of boxes in each row. I.e, the number of boxes in each row is smaller than or equal to the number of boxes in the row above it.*

Example 1.1. Some examples

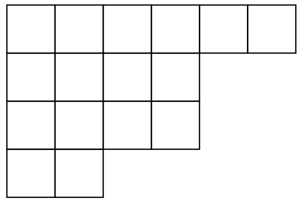


Its easy to see that if n is the total number of boxes in a young diagram and λ_i is the number of boxes in row i, then $\lambda = (\lambda_1, \dots, \lambda_m)$ is a partition of n. Conversely, given a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ of n (where $\lambda_1 \geq \cdots \geq \lambda_m$) we can obtain a young diagram with row i having λ_i boxes. This naturally gives a bijection

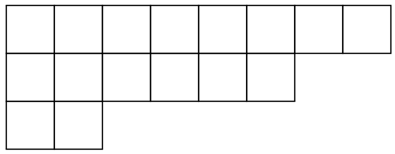
$$\{\text{partitions } \lambda \text{ of } n\} \longleftrightarrow \{\text{young diagrams with } n \text{ boxes}\}.$$

Notation 1.2. When we write $\lambda \vdash n$ we mean that λ is a partition of n. Also given a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ we write $|\lambda|$ to denote the number being partitioned. So for example if $\lambda = (4, 2, 1)$ then $|\lambda| = 7$ and if $\mu = (3)$ then $|\mu| = 3$.

Example 1.3. The integer $n = 16$ can be partitioned as $\lambda = (6, 4, 4, 2)$ which corresponds to the young diagram



It could also be partitioned as $\mu = (8, 6, 2)$ which corresponds to the young diagram



Definition 2: *Any way of putting positive integers in each box of a young diagram is a* ***numbering*** *or* ***filling*** *(the former requires distinct entries whereas the latter does not). A* ***young tableau*** *is a filling of a young diagram that is*

(1) weakly increasing across each row, and
(2) strictly increasing down each column.

We say that a young diagram λ is ***the shape of the tableau****.*

Example 1.4. Some examples:

1	2	3	4
5	6		
7			

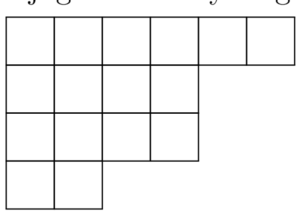
1	2	2	3	3	5
2	3	5			
4	4	6			
5	6				

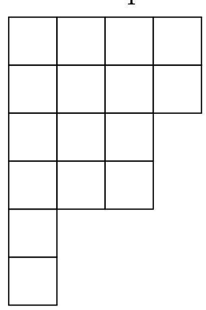
1	3	7	12	13	15
2	5	10	14		
4	8	11	16		
6	9				

The last young diagram above is formally known as a **standard tableau**; a tableau in which the entries are the integers $1, \ldots, n$ each occurring once.

Definition 3: *The* ***conjugate diagram*** $\widetilde{\lambda}$ *of a young diagram λ is the diagram obtained by flipping λ on its main diagonal (i.e $y = -x$). As a partition, it describes the lengths of the columns of λ.*

Example 1.5. The conjugate of the young diagram corresponding to $\lambda = (6, 4, 4, 2)$ is





Definition 4: *Any filling T of a young diagram determines (gives) a filling of the conjugate diagram denoted T^{τ}. We call the filling of a conjugate diagram the* ***transpose****.*

Remark 1.6. The transpose of a standard tableau is a standard tableau, however the same is **not true** for taking a transpose of a young tableau.

Consider a filling T of a young diagram corresponding to $\lambda = (6, 4, 4, 2)$

1	2	2	3	3	5
2	3	5	5		
4	4	6	6		
5	6				

.

We can construct a monomial x^T as follows: for each $1 \leq i \leq 4$ we let e_i denote the number of times that i appears in the filling T. Then set $x^T \coloneqq \prod_{i=1}^{4} x_i^{e_i} = x_1 x_2^3 x_3^3 x_4^2 x_5^4 x_6^3$. More generally,

Definition 5: *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition and T a filling of the young diagram corresponding to λ. For each $1 \leq i \leq m$ let e_i denote the number of times that i appears in T. Then we define a monomial $x^T \coloneqq \prod_{i=1}^{m} x_i^{e_i}$. The **Schur polynomial** $s_\lambda(x_1, \dots, x_m)$ is the sum*

$$s_\lambda(x_1, \dots, x_m) = \sum_T x^T,$$

where T ranges over all young tableau's with shape λ using the numbers 1 to m.

Example 1.7. Consider the young diagram corresponding to the partition $\lambda = (n)$ which is simply a single row with n boxes. How many tableaus T with with shape λ using the numbers 1 to n are there? This question is equivalent to asking what are all the possible monomials in a homogeneous polynomial of degree n in n variables. Recall that there are $\binom{n+d-1}{d}$ possible monomials of degree d in n variables. Thus there are $\binom{2n-1}{n}$ possible tableaus T with shape λ using the numbers 1 to n. For $n = 3$, this gives $\binom{2n-1}{n} = 10$ possible options. The corresponding Schur polynomial is

$$\begin{aligned} h_3(x_1, \dots, x_n) &= \sum_{\substack{a_1 + \cdots + a_n = 3 \\ a_i \geq 0}} x_1^{a_1} \cdots x_n^{a_n} \\ &= x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2 + x_1 x_2 x_3.. \end{aligned}$$

Using this simple example, we can make the following conclusions.

Theorem 1.8. *The n^{th} **complete symmetric polynomial** is the sum of all distinct monomials of degree n in m variables. It is denoted $h_n(x_1, \dots, x_m)$. Then for the young diagram corresponding to $\lambda = (n)$, the Schur polynomial is $s_\lambda(x_1, \dots, x_m) = h_n(x_1, \dots, x_m)$.*

On the other hand, if we consider the young diagram corresponding to $\lambda = (1, \dots, 1)$ (appearing n times) then the Schur polynomial $s_\lambda(x_1, \dots, x_m)$ in this case is precisley the n^{th} elementary symmetric polynomial in m variables; that is the polynomial $e_n(x_1, \dots, x_m)$ which defined as the sum of all monomials $x_{i_1} \cdots x_{i_n}$ for all increasing sequences $1 \leq i_1 < \cdots < i_n \leq m$.

Definition 6: *A **skew diagram** or **skew shape** is the diagram obtained by removing a smaller young diagram from a larger one that contains it. If $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_k)$ are partitions, we write $\mu \subseteq \lambda$ if the young diagram of μ is contained* *in the young diagram of λ. Equivalently, if $\mu_i \leq \lambda_i$ for all i. The resulting skew shape is denoted $\lambda \setminus \mu$.*

A ***skew tableau*** *is a filling of the boxes of a skew diagram with positive integers, weakly increasing in rows and strictly increasing down columns. The diagram is called its* ***shape****.*

Example 1.9. Consider the young diagram corresponding to $\lambda = (5, 5, 4, 3, 2)$, and let $\mu = (4, 4, 1)$. Below we embed the young diagram corresponding to μ inside λ with $\star$'s. Right next to it is the skew diagram $\lambda \setminus \mu$ and finally a skew tableau having the skew diagram before it as its shape:

$\star$	$\star$	$\star$	$\star$	$\star$	
$\star$	$\star$	$\star$	$\star$	$\star$	
$\star$					

$\times$	$\times$	$\times$	$\times$	$\times$	
$\times$	$\times$	$\times$	$\times$	$\times$	
$\times$					

$\times$	$\times$	$\times$	$\times$	$\times$	2
$\times$	$\times$	$\times$	$\times$	$\times$	6
$\times$	2	4	4		
2	3	6			
5	6				

.

1.2. **Row-Insertion.** For a positive integer $x \in \mathbb{N}$ and a Young tableau T, we wish to construct a young tableau having entries exactly of T along with one more entry containing x. We describe an algorithm to do so. Before doing so, we introduce some notation: for a filling T of a young diagram, we let $T_{i,j}$ denote the positive integer in $B_{i,j}$: the box located at the i^{th} row and j^{th} column.

Algorithm 1.10. *Given a tableau T with shape $\lambda = (\lambda_1, \ldots, \lambda_m)$ and a positive integer $x \in \mathbb{N}$, construct a new tableau $T \leftarrow x$ as follows:*

(1) If $x \geq T_{1,j}$ for all $1 \leq j \leq \lambda_1$, then simply add a new box containing x at the end of the first row.
(2) If not, find the smallest j for which $T_{1,j} > x$, remove (or ***bump)*** *$T_{1,j}$ and replace it with x.*
(3) Repeat procedure on the next row this time using $T_{1,j}$.
(4) Continue until a new box is added.

Example 1.11. Insert $x = 4$ into the following tableau with filling T:

1	2	5	8
3	4	7	
6			

Solution: The solution is to preform the following "bumps":

1	2	4	8
3	4	7	
6			

1	2	4	8
3	4	5	
6			

1	2	4	8
3	4	5	
6	7		

Example 1.12. Insert $x = 5$ into the following tableau with filling T:

1	1	2	3	4	6
2	3	3	5		
4	5	6			
7					

Solution: Preform the following "bumps":

1	1	2	3	4	5
2	3	3	5		
4	5	6			
7					

1	1	2	3	4	5
2	3	3	5	6	
4	5	6			
7					

Example 1.13. Insert $x = 2$ into the following tableau with filling T:

1	2	2	3
2	3	5	5
4	4	6	
5	6		

Solution: Preform the following "bumps":

1	2	2	2
2	3	5	5
4	4	6	
5	6		

1	2	2	2
2	3	3	5
4	4	6	
5	6		

1	2	2	2
2	3	3	5
4	4	5	
5	6		

1	2	2	2
2	3	3	5
4	4	5	
5	6	6	

Definition 7: *A row-insertion of $T \leftarrow x$ determines a collection $\mathcal{R} = \{B_{i,j}\}$ of boxes in which elements are "bumped" during the algorithm along with the **new box** that is added at the end. The set $\mathcal{R}$ is called the **bumping route**. Given two bumping routes $\mathcal{R}$ and $\mathcal{R}'$, we say that $\mathcal{R}$ is **strictly left** (resp. **weakly left**) if whenever $B_{i,j} \in R$, $B_{i,j-1} \in \mathcal{R}'$ (resp. $B_{i,j-1}$ or $B_{i,j}$ is in $\mathcal{R}'$).*

Example 1.14. For Example 1.11, the bumping route is $\mathcal{R} = \{B_{1,3}, B_{2,3}, B_{3,2}\}$. For Example 1.12, the bumping route is $\mathcal{R} = \{B_{1,6,}, B_{2,5}\}$. For Example 1.13, the bumping route is $\mathcal{R} = \{B_{1,4}, B_{2,3}, B_{3,3}, B_{4,3}\}$.

Lemma 1.15 (Row-Bumping)**.** *Consider two successive row-insertions, first $T \leftarrow x$ and then $(T \leftarrow x) \leftarrow x'$; i.e row-inserting x' into the tableau obtained from $T \leftarrow x$. The first gives rise to a bumping route $\mathcal{R}$ along with a new box $B_{k,\ell}$, and the second gives rise to a bumping route $\mathcal{R}'$ along with a new box $B_{k',\ell'}$.*

(1) If $x \leq x'$, then R is strictly left of R' and $\ell < \ell'$ and $k \leq k'$.
(2) If $x > x'$, then R' is weakly left of R and $\ell' \leq \ell$ and $k' < k$.

We now describe an algorithm for forming the product tableau of two young tableaus T and U:

Algorithm 1.16. *Given two young tableaus T and U, form the product young tableau $T \cdot U$ as follows: assume the shape of T is $\lambda = (\lambda_1, \ldots, \lambda_n)$ and the shape of U is $\mu = (\mu_1, \ldots, \mu_m)$, then*

(1) Compute the tableau $(T \leftarrow U_{m,1}) \leftarrow U_{m,2} \leftarrow \cdots \leftarrow U_{m,\mu_m}$. Let A_1 be the tableau obtained.
(2) Compute $(A \leftarrow U_{m-1,1}) \leftarrow U_{m-1,2} \leftarrow \cdots \leftarrow U_{m-1,\lambda_{m-1}}$. Let A_2 be the tableau obtained.
(3) Continue until tableau A_m is obtained and set $T \cdot U = A_m$.

Claim 1. *The product operation makes the set of tableaux into an associative monoid. The empty tableau $\varnothing$ is a unit in this monoid: $\varnothing \cdot T = T \cdot \varnothing = T$.*

2. Lecture 2 - Wednesday September 24

Start with notation convention for young diagram and fillings. Then mention convention of skew diagrams with $\times$

Definition 8: *Let $S = \lambda \setminus \mu$ be a skew diagram. An **inside corner** is a box $S_{i,j}$ in μ such that $S_{i+1,j}$ and $S_{i,j+1}$ are not in λ. An **outside corner** is a box $S_{k,\ell}$ in λ such that $S_{k+1,\ell}$ and $S_{k,\ell+1}$ are not in λ.*

Example 1.17. Consider the following skew diagram S:

×	×	×	
×	×	×	
×			

The inside corners are boxes $S_{3,1}$ and $S_{2,3}$. The outside corners are the last boxes in the second, third and fourth row.

Given a skew tableau $S = \lambda \setminus \mu$, there is an algorithm that removes any inside corners in S. Moreover, this process preserves the property of being tableau: i.e the resulting diagram is a tableau.

Algorithm 1.18. *Let $S = \lambda \setminus \mu$ be a skew tableau. Suppose $S_{k,\ell}$ is an inside corner.*

*(1) If $S_{k+1,\ell} \leq S_{k,\ell+1}$; then **swap** $S_{k,\ell}$ and $S_{k+1,\ell}$; i.e set $S_{k,\ell} := S_{k+1,\ell}$ and make $S_{k,\ell}$ the **empty box** (the box with no entry).*
(2) If $S_{k+1,\ell} > S_{k,\ell+1}$, then swap $S_{k,\ell}$ and $S_{k,\ell+1}$.
(3) Repeat steps (1) and (2) on the resulting empty box. Stop if the empty box obtained is an outside corner.
(4) Remove the empty box from the diagram.

Example 1.19. Let S be the following skew tableau:

			5
		2	8
4	5	7	

The following steps remove the inside corner $S_{2,2}$:

			5
		2	8
4	5	7	

			5
	2		8
4	5	7	

			5
	2	7	8
4	5		

			5
	2	7	8
4	5		

Remark 1.20. A skew tableau $S = \lambda \setminus \mu$ which has no inside corners *is a tableau*; i.e $\mu = \varnothing$ is the empty tableau (necessarily). So the process of removing all inside corners from a skew tableau is a process of transforming (or "rectifying") S into a tableau.

Definition 9: *Let S be a skew tableau. The tableau obtained by successively removing all inside corners from S using Algorithm 1.18 above is called **the rectification** of S, and is denoted* $\operatorname{Rect}(S)$. *The process of successively removing inside corners is known as **jeu de taquin** (which means "teaser game").*

Claim 2. *Starting with a given skew tableau, any sequence of inside corners chosen during the process of rectifying lead to the same rectification of S.*

2.1. **Another product of young tableaus.** We give another way of constructing a product of two young tableaus using rectification.

Definition 10: *Let T and U be young tableaus. Take a rectangle of empty squares with the same number of columns as T and the same number of rows as U. Attach T below this rectangle and U to the right. The resulting skew tableau is denoted $T * U$. Define the **product** $T \cdot U$ of T and U to be the rectification* $\operatorname{Rect}(T * U)$.

Claim 3. *This product agrees with the first definition. In other words, the resulting tableau is the same regardless of the product operation used.*

Exercise 1: Compute the product $T * U$

T =

1	2	2	3
2	3	5	5
4	4	6	
5	6		

U =

1	3
2	

Exercise 2: The product $*$ is associative.

2.2. **Chapter 2 : Words; The plactic monoid.**

Definition 11: *A **word** (for us) will be a string of integers. The integers in a word are called **letters**. Given a tableau or skew tableau T, we define the **word of** T, denoted $w(T)$, to be the word obtained by reading the entries of T starting at the bottom row, left to right. To be more precise, if the shape of T is $\lambda = (\lambda_1, \dots, \lambda_n)$ then*

$$w(T) \coloneqq T_{n,1}T_{n,2}\cdots T_{n,\lambda_n}T_{n-1,1}T_{n-1,2}\cdots T_{n-1,\lambda_{n-1}}\dots T_{1,1}\cdots T_{1,\lambda_1}.$$

Remark 1.21. If T is a tableau with shape λ, then $w(T)$ will have $|\lambda|$ words.

Example 1.22. Consider the tableau

T =

1	2	2	5
4	5	7	
6	8		

Then the word is $w(T) = 684571225$.

On the other hand, given a tableau U with shape μ and its word $w(U) = w_1 w_2 \cdots w_{|\mu|}$ we can recover its diagram as follows : whenever there are indices $i < j$ for which $w_i > w_j$, place a bar between the integers w_i and w_j: $w(U) = w_1 w_2 \cdots w_i | w_j w_{j+1} \cdots w_{|\mu|}$. Strings of integers between bars correspond to the rows of the young diagram (starting from the bottom row).

Example 1.23. Suppose U is a tableau with word $w(U) = 794571235$. Recover its diagram.

Solution: The diagram is

U =

1	2	3	5
4	5	7	
7	9		

2.3. Elementary Knuth Transformations.

Definition 12: *We define two transformations on three letter words:*

$$yzx \mapsto yxz \quad \text{if } x < y \leq z \tag{K'}$$

$$xzy \mapsto zxy \quad \text{if } x \leq y < z. \tag{K''}$$

The transformations K',K'',$(K')^{-1}$ and $(K'')^{-1}$ are known as ***elementary Knuth transformations****. Two words w, w' are* ***Knuth equivalent*** *if one can be changed into the other by preforming a sequence of elementary Knuth transformations. In this case we write $w \equiv w'$.*

Example 1.24. Prove that the words $w = 13452$ and $w' = 31245$ are equivalent.

Solution: Preform the following transformations:

$$\begin{aligned} 13452 &\xrightarrow{K''} 13425 \\ &\xrightarrow{K''} 13245 \\ &\xrightarrow{K'} 31245. \end{aligned}$$

We can achieve the row-inserting algorithm introduced in the previous lecture by continuously preforming elementary Knuth transformations. More precisely, if $w(T)$ is the word of a tableau T and we wish to row-insert x into T, then we continuously preform elementary Knuth transformations on the word $w(T)x$, and then read off the resulting tableau. In other words,

Proposition 1.25. *For any tableau T and $x \in \mathbb{N}$,*

$$w(T \leftarrow x) \equiv w(T)x.$$

Example 1.26. For the young tableau

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 4 \\ \hline 3 & 5 & 5 \\ \cline{1-3} 7 & 8 \\ \cline{1-2} \end{array}$$

the word is $w(T) = 783551134$. Let us row insert $x = 3$:

$$\begin{aligned} w(T)3 &= 7835511343 && (K'') \\ &= 7835511433 && (K'') \\ &= 7835514133 && (K'') \\ &= 7835541133 && (K') \\ &= 7835451133 && (K'') \\ &= 7853451133 && (K') \\ &= 7583451133. \end{aligned}$$

The corresponding diagram is $(w(T)3 = 7|58|345|1133)$:

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 3 & 4 & 5 \\ \cline{1-3} 5 & 8 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}$$

Corollary 1.27. *Given tableaux T and U,*

$$w(T \cdot U) \equiv w(T)w(U).$$