CONNECTED AND PATH CONNECTED SPACES

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1. Introduction

We will explore the idea of *connectedness* and *path connectedness* over abstract metric spaces (X, d). It is a well known fact that given any metric space X, path-connectedness immediately implies connectedness (in fact this is true over any topological space as well). However, the converse does not always hold. The first main result is to prove the forward direction, and the second main result is to prove the converse direction, though only under certain restrictions on our space and set in question.

2. Part I - Connectedness

Definition 1.1. Let X be a metric space and $U \subseteq X$. We say that the sets $V, W \subseteq X$ **separate** U in X if

$$V \cap U \neq \emptyset, \;\; W \cap U \neq \emptyset, \;\; V \cap W = \emptyset, \;\; U \subseteq V \cup W. \tag{1}$$

We say that U is **connected** in X if there do not exist open sets $V, W \subseteq X$ that separate U in X.

Remark 1.2. When $U = X$ in Definition 1.1 then we have that a metric space X is connected if and only if there do not exist non-empty disjoint open sets $V, W \subseteq X$ such that $X = V \cup W$. The following proposition gives us an alternative test for the connectivity of a metric space.

Proposition 1.3. *Let X be a metric space. Then X is connected if and only if the only subsets of X that are both open and closed are X and $\emptyset$.*

Proof. First, suppose that X is connected. Let $U \subsetneq X$ be a proper non-empty subset. Assume towards a contradiction that U is both closed and open. Then U and $V := U^c$ admit a separation of X, a contradiction. Conversely, suppose that the only subsets of X that are both open and closed are X and $\emptyset$. Assume towards a contradiction that X is not connected. Then $X = U \cup V$ for some non-empty disjoint open sets U, V. We emphasize that the conditions that U, V are disjoint and non-empty immediately imply that neither of them are the entire space X. We claim that $U^c = V$. Indeed, if $u \in U^c$ then $u \in X$ and $u \notin U$. The latter implies that $u \in U \cup V$, but then the former implies that $u \in V$. Conversely, if $v \in V$ then since U and V are disjoint, we must have $v \notin U$ so that $v \in U^c$. Therefore $U^c = V$. But then V is both a closed and an open set, a contradiction. $\square$

Example 1.4. We claim that the only non-empty connected sets of $\mathbb{Q}$ are the singleton sets. Indeed, let $\emptyset \neq U \subseteq \mathbb{Q}$. Assuming that U is not the singleton set, we have that U contains at least two distinct rational numbers a, b. Choose some irrational number $q \in \mathbb{R}$ such that $a < q < b$. Let $A = \{r \in U : r < q\}$ and $B = \{t \in U : t > q\}$. Then A and B are disjoint non-empty sets ($a \in A$, $b \in B$) that separate U. So U is not connected. On the other hand, if U was a singleton set then it is trivially connected.

Recall that in Theorem 8.5 in Lecture Shell 8 from [4] that for metric spaces X and Y, a function $f : X \to Y$ is continuous if and only if for all open subsets $V \subseteq Y$ we have that $f^{-1}(V)$ is an open subset of X. This gave as an equivalent definition of continuity which was contrary to the one stated in Definition 8.1 of the same Lecture Shell. This equivalence is in fact our notion of continuity in general topological spaces. In Proposition 15.1 in Lecture Shell 15 from [4], we showed that the image of a compact metric space under a continuous map is compact. We have a similar result for connectedness.

Proposition 1.5. *The image of a connected metric space under a continuous map is connected.*

Proof. Let X, Y be metric spaces. Let $f : X \to Y$ be a continuous map. Suppose that X is connected. Suppose for a contradiction that $f(X)$ is not connected. Let U, V be disjoint non-empty open sets in $f(X)$ that admit a separation. Then $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ must be a disjoint union of sets in X. Indeed, suppose not. Let $a \in f^{-1}(U) \cap f^{-1}(V)$ and suppose that $f(a) = b$. Then $f(a) = b \in U$ and $f(a) = b \in$, so that $U \cap V \neq \emptyset$, a contradiction. Our previous claim would then imply that $f^{-1}(U)$ and $f^{-1}(V)$ admit a separation of X, a contradiction. It follows that $f(Y)$ is a connected space. □

Theorem 1.6. *Let X be a metric space. The union of any set of connected sets in X, which share a common point, is connected.*

Proof. Let $\emptyset \neq K$ be any index set. For each $k \in K$, let $U_k \subseteq X$ be a connected set. Let $a \in X$ be so that $a \in \bigcap_{k \in K} U_k$ and $B = \bigcup_{k \in K} U_k$. Assume towards a contradiction that B is not connected. Then there exists non-empty open sets $V, W \subseteq X$ such that $V \cap B \neq \emptyset$, $W \cap B \neq \emptyset$ and $B \subseteq V \cup W$. We claim that there must exist some index $l \in K$ such that $U_l \subseteq V$. Indeed, since $V \cap B \neq \emptyset$, there must exist some $p \in B \cap V$. This implies that $p \in U_l$, for some $l \in K$, and $p \in V$ so that $U_l \cap V \neq \emptyset$. But then we must have that $U_l \subseteq V$. If not, then for some $u \in U_l$ we have that $u \notin V$. But then since $U_l \subseteq V \cup W$, we must have that $u \in W$. This implies that $W \cap U_l \neq \emptyset$ as well, which implies that V and W separate U_l, a contradiction. We conclude that $U_l \subseteq V$. A similar argument can be made to show that for some index $l \neq m \in K$, we must have that $U_m \subseteq W$ (note that we are assuming that K has at least two distinct elements since the result is otherwise trivial). But then the fact that $a \in U_m \cap U_l$ implies that $V \cap W \neq \emptyset$, a contradiction. We conclude that B is indeed connected. □

3. Part II - Path Connectedness

Given a metric space X, the notion of path connectedness is to say that any two points $a, b \in X$ lie in the image of some continuous map. The image of this map is what we usually refer to as a path. In turns out that in any metric space X, path connectedness implies connectedness. However, the converse, as we will see, does not hold in general.

Definition 1.7. Let X be a metric space and $A \subseteq X$. For $a, b \in A$, a **path** from a to b is a continuous map $f\colon [0,1] \to A$ such that $f(0) = a$ and $f(1) = b$. If there exists a path from a to b we will write $a \sim b$. If for all $a, b \in A$, we have that $a \sim b$ then we will say that A is **path connected.**

Given a normed linear space X, we recall from MATH 247 that for set $A \subseteq X$ and two points $a, b \in A$, the *line segment* between a and b is the set $[a, b] = \{a+t(b-a)\colon 0 \leq t \leq 1\}$. We also recall that we A is said to be *convex* if for every $a, b \in A$, $[a, b] \subseteq A$. Given the definition of path connectedness in Definition 1.7 it is immediately clear that if a set $A \subseteq X$ is convex then it is path connected.

Lemma 1.8. *Let X be a normed linear space and $a \in X$, $r > 0$ be arbitrary. Then $B(a, r)$ is a convex set.*

Proof. Let $x, y \in B(a, r)$. Let $f\colon [0,1] \to X$ be the map given by $f(t) = x + t(y - x)$. We wish to show that for all $t \in [0,1]$, $f(t) \in B(a, r)$. Using a well known trick ([1]) gives

$$
\begin{aligned}
f(t) - a &= x + t(y - x) - a \\
&= (1 - t)x + ty - a \\
&= (1 - t)x + ty + ta - ta - a \\
&= (1 - t)x - (1 - t)a + t(y - a) \\
&= (1 - t)(x - a) + t(y - a).
\end{aligned}
$$

Hence, using the fact that $\|x - a\| < r$ and $\|y - a\| < r$ gives

$$
\begin{aligned}
\|f(t) - a\| = \|(1 - t)(x - a) + t(y - a)\| &\leq \|(1 - t)(x - a)\| + \|t(y - a)\| \\
&= (1 - t)\|x - a\| + t\|y - a\| \\
&< (1 - t)r + tr \\
&= r.
\end{aligned}
$$

Hence $f(t) \in B(a, r)$, so that $B(a, r)$ is a convex set. □

4. Part III - Connection between connectedness and path connectedness

We are now ready to state the connection between connectedness and path connectedness. Before doing so, we will establish some common notation and conventions used when working with paths over general topological spaces. We then state a lemma related to a topological characterization of continuity. Finally, we prove our main results.

Definition 1.9. Let X be a metric space and $a, b, c \in X$. We let $\kappa_a(t)$ be the map $\kappa_a\colon [0,1] \to X$ given by $\kappa_a(t) = a$. We refer to κ_a as the **constant path** at a in X.

When γ is a path from a to b in X, we let γ^{-1} be the map $\gamma^{-1}\colon [0,1] \to X$ given by $\gamma^{-1}(t) = \gamma(1-t)$. We refer to γ^{-1} as the **inverse path** from a to b in X. Finally, when α is a path from a to b in X and β is a path from b to c in X, we let $\alpha \cdot \beta(t)$ be the map $\alpha \cdot \beta(t)\colon [0,1] \to X$ given by

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t-1) & 1/2 \leq t \leq 1 \end{cases}.$$

We refer to $\alpha \cdot \beta(t)$ as the **product path** (or the **concatenation path**) from a to c in X.

Lemma 1.10. *Let X be a metric space and $G \subseteq X$. Let $f\colon G \to X$ be a continuous function. Then for each open set $V \subseteq X$, there exists an open set $U \subseteq X$ such that $f^{-1}(V) = G \cap U$.*

Proof. Let $V \subseteq X$ be an open set. Let $w \in f^{-1}(V)$, so that $f(w) \in V$. Since V is an open set, there exists some $r_w > 0$ such that $B(f(w), r_w) \subseteq V$. Since f is continuous on G, for r_w there exists a corresponding $\delta_w > 0$ such that for all $z \in G$, $z \in B(w, \delta_w)$ implies that $f(z) \in B(f(w), r_w) \subseteq V$. So in fact $B(w, \delta_w) \cap G \subseteq f^{-1}(V)$. Hence, let

$$U = \bigcup_{w \in f^{-1}(V)} B(w, \delta_w),$$

where each $\delta_w > 0$ corresponds to each $r_w > 0$ such that $B(f(w), r_w) \subseteq V$ (this is a similar construction as above). We claim that $f^{-1}(V) = G \cap U$. Indeed, for any $v \in f^{-1}(V)$, $v \in G$ be definition and $v \in B(v, \delta_v)$, so $f^{-1}(V) \subseteq G \cap U$. For the converse direction, our argument is similar from before, namely if $v \in G \cap U$, then $v \in G$ and $v \in B(z, \delta_z)$ for some $z \in f^{-1}(V)$. But then by the definition of δ_z, we have that $v \in B(z, \delta_z)$ implies that $f(v) \in B(f(z), r_z) \subseteq V$. So $v \in f^{-1}(V)$. □

We are now ready to state the first of the two results.

Theorem 1.11. *Let X be a metric space and $U \subseteq X$. If U is path connected then U is connected.*

Proof. Assume towards a contradiction that U is not connected. Then there exists open sets $V, W \subseteq X$ such that $U = V \cup W$ and $V \cap W = \emptyset$. Let $z_0 \in V$ and $z_1 \in W$, $\gamma : [0,1] \to U$ be a continuous path between them such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$, and let

$$t^* = \inf\{t \in [0,1] \colon \gamma(t) \in W\}.$$

We remark that since V and W are disjoint, it must be the case that $\gamma^{-1}(V) \cup \gamma^{-1}(W)$ is disjoint as well. Indeed, if not then for some $w \in \gamma^{-1}(V) \cap \gamma^{-1}(W)$ we have that $\gamma(w) \in V$ and $\gamma(w) \in W$, which contradicts the fact that W and V are disjoint. We also remark that $t^* \in [0,1]$. By Lemma 1.10, since γ is continuous, there exists open sets $F, E \subseteq \mathbb{R}$ such that $\gamma^{-1}(W) = [0,1] \cap F$ and $\gamma^{-1}(V) = [0,1] \cap E$. We proceed with a case analysis.

First, suppose that $t^* \in \gamma^{-1}(W)$. Then $t^* \in F$ and note that $t^* > 0$ since if $t^* = 0$ then $t^* \in \gamma^{-1}(V)$, contradicting the fact that $\gamma^{-1}(W)$ and $\gamma^{-1}(V)$ are disjoint. Since

F is open, $D(t^*, \delta) \subseteq F$ for some $\delta > 0$. Intersecting this with $[0,1]$ implies that $(t^* - \delta, t^*] \subseteq \gamma^{-1}(W)$. Indeed, since $0 \notin F$ we must have that $t^* - \delta \geq 0$ and so for any $r \in (t^* - \delta, t^*]$, we have $r \in [0,1]$ and $r \in F$ which gives $r \in \gamma^{-1}(W)$. Hence $(t^* - \delta, t^*] \subseteq \gamma^{-1}(W)$. But then for any $r' \in (t^* - \delta, t^*)$, $\gamma(r') \in W$ and $r' < t^*$, which contradicts the definition of t^*.

Now suppose that $t^* \in \gamma^{-1}(V)$. Then $t^* \in E$ and note that $t^* < 1$ since if $t^* = 1$ then $t^* \in \gamma^{-1}(W)$, contradicting the fact that $\gamma^{-1}(W)$ and $\gamma^{-1}(V)$ are disjoint. Since E is open, $D(t^*, \delta) \subseteq E$ for some $\delta > 0$. Intersecting this with $[0,1]$ implies that $[t^*, \delta + t^*) \subseteq \gamma^{-1}(V)$. Indeed, since $1 \notin E$ we must have $t^* + \delta \leq 1$ and so for any $r \in [t^*, \delta + t^*)$, we have $r \in [0,1]$ and $r \in E$ which gives $r \in \gamma^{-1}(V)$. Hence $[t^*, \delta + t^*) \subseteq \gamma^{-1}(V)$. But then by the definition of t^*, there exists a $r' \in \gamma^{-1}(W)$ such that $r' \in [t^*, \delta + t^*) \subseteq \gamma^{-1}(V)$, which contradicts the fact that these two sets are disjoint. In both cases, we derived a contradiction and hence we conclude that U must be connected. □

Unfortunately, the converse of Theorem 1.11 is not always true in the metric space setting. A famous example is the *topolgist sine curve* (see [3] and also [5] for a more detailed discussion). In short, if we let $A = \{(x, \sin(1/x)) \colon x > 0\}$ then the closure of A is precisely $\mathrm{cl}(A) = A \cup B$ where $B = \{(0, y) \colon y \in [-1,1]\}$ is the set of limit points of A. One can show that $\mathrm{cl}(A)$ is connected but fails to be path-connected. However, if our setting is a normed linear space X and if we consider an *open set* $U \subseteq X$ specifically, then the converse does hold. We prove it below.

Theorem 1.12. *Let X be a normed linear space and $U \subseteq X$ be an open set. If U is connected then U is also path-connected.*

Proof. Assume towards a contradiction that U is not path connected. Then there exists some $z_0, z_1 \in U$ such that there does not exist a path from z_0 to z_1 in U. Consider the path components of U:

$$V_z = \{z' \in U : \exists \text{ a continuous path from } z \text{ to } z'\}.$$

We show that V_z is open for any $z \in U$. Let $w \in V_z$. Since U is open, $B(w, r) \subseteq U$ for some $r > 0$. Let $v \in B(w, r)$. By Lemma 1.8, we have that $B(w, r)$ is a convex set and hence the map $\gamma' : [0,1] \to B(w, r)$ given by $\gamma'(t) = w + t(v - w)$ is a continuous path from w to v in $B(w, r)$. Since $w \in V_z$, there exists a continuous path γ from z to w in U. By a concatenation of paths, we may take the path $f = \gamma \cdot \gamma'(t)$ to get a continuous path from z to v in U, and so $v \in V_z$. Hence $B(w, r) \subseteq V_z$ so that V_z an open set. We claim that V_{z_0} and V_{z_1} are disjoint. Indeed, if not then there would be some $v \in U$ such that $v \in V_{z_0}$ and $v \in V_{z_1}$. This would imply that there exists continuous paths $\gamma, \gamma' : [0,1] \to U$ from z_0 to v and from z_1 to v, respectively. But then the path $\gamma \cdot \gamma'^{-1}(t)$ would be a continuous path from z_0 to z_1, contradicting our assumption that there does not exist a path from z_0 to z_1 in U. In fact by a similar argument, for any $z, w \in U$, if $V_w \neq V_z$ then $V_w \cap V_z = \emptyset$. Of course for any $w \in U$, $w \in V_w$ since the constant path $\kappa(t) = w$ is a continuous path from w to itself. This in fact implies that the connected components of U are open and form a partition on U and hence separate it, contradicting the fact that U is connected. □

References

[1] Stephen New (2024) *MATH 247 Calculus 3 (Advanced) Lecture Notes*, University of Waterloo. Found at: https://www.math.uwaterloo.ca/~snew/math247-2024-W/index.html

[2] Stephen New (2023) *PMATH 467 Algebraic Topology Lecture Notes*, University of Waterloo. Found at: https://www.math.uwaterloo.ca/~snew/pmath467-2023-W/LectureNotes/index.html

[3] Stephen New (2022) *PMATH 351 Real Analysis Lecture Notes*, University of Waterloo. Found at: https://www.math.uwaterloo.ca/~snew/pmath351-2022-S/LectureNotes/index.html

[4] A. Nica. Lecture shells for PMath 351 lectures in Winter Term 2024, available on the Learn web-site of the course.

[5] Keith Conrad *Spaces that are connected but not path-connected*, University of Conetecut. Found at: https://kconrad.math.uconn.edu/blurbs/topology/connnotpathconn.pdf